BOUNDARY VALUE AND EXPANSION PROBLEMS: OSCILLATION, COMPARISON AND EXPANSION THEOREMS.*

By R. D. Carmichael.

1. *Algebraic Oscillation and Comparison Theorems.*—On any convenient horizontal straight line segment, say the points s such that $a \leqq s \leqq b$, let us erect n perpendiculars two of which are at the ends of the segment while the other $n - 2$ are evenly or unevenly distributed on the interior of the segment. Let these be marked from left to right by the numbers $1, 2, \cdots, n$; and consider them as analogous to the n coördinate axes of a space of n dimensions. Let the greatest distance between two consecutive axes be called the norm of the system of axes. Having given the set of real constants $u_1, u_2, \cdots, u_n$, let us take a point on the ith axis at a distance $|u_i|$ from the original segment and above it or below it according as u_i is positive or negative. Having done this for each value i of the set $1, 2, \cdots, n$, join by straight line segments the point on each interior axis to the points on the two adjacent axes. We thus obtain a broken line which we shall call the graphic representation of the point $(u_1, u_2, \cdots, u_n)$ in space of n dimensions or of the set of constants $u_1, u_2, \cdots, u_n$. This broken line is the graph of a continuous function $u(s)$ of the real variable s on the interval $a \leqq s \leqq b$. We shall say that this function $u(s)$ is obtained from the set of constants $u_1, u_2, \cdots, u_n$ by *linear interpolation* with respect to the given n axes. The zeros of $u(s)$ we shall call the *zeros* of the set of constants with respect to the given system of coördinates.

Let us consider the system of n equations

$$\sum_{j=1}^{n+2} a_{ij}x_j = 0, \qquad i = 1, 2, \cdots, n, \tag{1.1}$$

in the $n + 2$ unknown quantities $x_1, x_2, \cdots, x_{n+2}$, the matrix of coefficients of this system being of rank n. Let D_i denote the determinant of the matrix obtained from the matrix of coefficients in (1.1) by striking out the ith and $(i + 1)$th columns. We then have the following fundamental theorem:†

Theorem I. *Let D_i for a given range R of consecutive values of the integer i be of one sign and let I denote the interval of the s-axis corresponding to this range of i in the sense of the first paragraph above. Let u_i and v_i be any two linearly independent solutions of the system* (1.1) *the matrix of whose coefficients is of rank n; and let these solutions be extended, by the method of*

* Presented to the American Mathematical Society.

† *American Journal of Mathematics*, 43 (1921): 69–101; see p. 84.

linear interpolation employed above, to the functions $u(s)$ and $v(s)$. Then on the interval I the zeros of $u(s)$ and $v(s)$ separate each other; that is, between any two consecutive zeros (on I) of one of these functions there is one and just one zero of the other function.

On examining the proof of this theorem, in the article cited, it is seen that the only use made of the hypothesis on D_i is in showing that ω_i,

$$\omega_i = \begin{vmatrix} u_i & v_i \\ u_{i+1} & v_{i+1} \end{vmatrix},$$

is of one sign in I, so that the theorem might be restated with D_i replaced by ω_i in the first line. The theorem as first stated is in the more useful form for suggesting analogous theorems in the transcendental cases; but the second form will be found more suggestive for proofs.

The foregoing theorem is analogous to the Sturmian theorem of oscillation for homogeneous linear differential equations of the second order and indeed reduces to that theorem by a certain limiting process. The object of this paper is (a) to derive (§ 1) the algebraic theorems which are analogous to the Sturmian theorems of comparison for homogeneous linear differential equations of the second order and of which the latter are limiting forms, (b) to obtain (§ 2) theorems of oscillation for differential equations of order n and for certain functional equations including difference and q-difference equations, (c) to derive (§ 3) corresponding theorems of comparisons by aid of the named algebraic theorems of comparison, (d) to point out (§ 4) a certain generalization of boundary conditions for expansion problems, and (e) to indicate (§ 5) the character of certain expansion problems for q-difference and integro-q-difference equations.

In what follows in this section we shall assume that the notation in (1.1) is so chosen that the determinant D_1 is different from zero. If the equations are taken in a suitable order (and we shall suppose them so written already), it is obviously possible to combine them into a new system having the same solutions x in such way that the new coefficients a_{ij} have the value zero when $j > i + 2$ and such that every $a_{i,i+2}$ is different from zero; and we reduce the latter to unity by dividing both members of the ith equation by $a_{i,i+2}$. We shall now suppose further that the original, and hence the new, matrix $||a_{ij}||$ has the property that the determinants of orders $1, 2, \cdots, n$ in its upper left-hand corner are all different from zero in value. Then it is possible to make further combinations of equations in the system so as to arrive at a new system with the same solutions x and of such sort that the new coefficients a_{ij} are zero when $j < i$. Then, changing the order of terms in the equation, we have a system in the form

$$(1.2) \qquad x_{i+2} + \alpha_i x_{i+1} + \beta_i x_i = 0, \qquad i = 1, 2, \cdots, n,$$

where α_i and β_i are determinate functions of the original coefficients a_{ij}.

On writing $x_i = y_i u_i$, the foregoing equation reduces to the following:

$$y_{i+2}u_{i+2} + \alpha_i y_{i+1}u_{i+1} + \beta_i y_i u_i = 0. \tag{1.3}$$

We set $y_{i+2} = \beta_i y_i$. If y_1 and y_2 are any two numbers different from zero, we have

$$y_{2i+1} = \beta_{2i-1}\beta_{2i-3}\cdots\beta_3\beta_1 y_1, \qquad y_{2i} = \beta_{2i-2}\beta_{2i-4}\cdots\beta_4\beta_2 y_2.$$

If β_i is positive for all i and if y_1 and y_2 are positive, it is clear that y_i is always positive and hence that u_i and x_i have always the same sign. Therefore the functions $x(s)$ and $u(s)$, obtained by linear interpolation from x_i and u_i with respect to a given system of coördinates, have their zeros on the same intervals of the coördinate system (though not necessarily at the same points). Equation (1.3) may be written in the form

$$u_{i+2} + \varphi_i u_{i+1} + u_i = 0, \qquad \varphi_i = \alpha_i \frac{y_{i+1}}{y_{i+2}}, \tag{1.4}$$

where φ_i is a determinate function of the a_{ij} in (1.1). From the principal properties of the distribution of the zeros of $u\,(s)$ one knows the principal properties of the distribution of the zeros of $x\,(s)$ provided that β_i is positive for each value of i.

A second equation of the form (1.1), under such hypotheses as we have just employed, would reduce to the normal form

$$v_{i+2} + \psi_i v_{i+1} + v_i = 0. \tag{1.5}$$

Comparison theorems for the distribution of the zeros of the functions $u(s)$ and $v(s)$, obtained from the constants u_i and v_i by linear interpolation, yield corresponding theorems for the two original systems of form (1.1). We shall derive and state the results only for the normal forms (1.4) and (1.5).

Multiplying (1.4) member by member by v_{i+1} and (1.5) by $-u_{i+1}$ and adding, we have a result which may be put in the form

$$\Delta\{u_{i+1}v_i - u_i v_{i+1}\} + (\varphi_i - \psi_i)u_{i+1}v_{i+1} = 0;$$

whence, on summing as to i from μ to m, it follows that

$$(u_{m+2}v_{m+1} - u_{m+1}v_{m+2}) - (u_{\mu+1}v_\mu - u_\mu v_{\mu+1}) + \sum_{i=\mu}^{m} (\varphi_i - \psi_i)u_{i+1}v_{i+1} = 0, \tag{1.6}$$

where μ and m are any two numbers of the set $1, 2, \cdots, n$ such that $\mu \leqq m$.

If solutions u_i and v_i of (1.4) and (1.5), respectively, interpolate into functions $u(s)$ and $v(s)$ having a common zero, say on the μth interval, and if $u(s)$ has a zero on an interval to the right of the μth, say on the

$(m+1)$th interval, m being chosen as small as possible, we may show that $v(s)$ has a zero between these two zeros of $u(s)$ provided that $\varphi_i \leqq \psi_i$, $i = \mu, \mu+1, \cdots, m$, the equality sign not holding for all these values. Without loss of generality in argument we take $u(s)$ and $v(s)$ to be positive each on the interval from its zero on the μth interval to its next zero to the right. We shall prove the statement in consideration by showing that we are led to a contradiction if we suppose that $v(s)$ is nowhere negative between the two consecutive zeros of $u(s)$. We have $u_{\mu+1}v_\mu - u_\mu v_{\mu+1} = 0$, since $u(s)$ and $v(s)$ have a common zero on the μth interval. Then from (1.6) we see that we now have

$$u_{m+2}v_{m+1} - u_{m+1}v_{m+2} = \sum_{i=\mu}^{m} (\psi_i - \varphi_i)u_{i+1}v_{i+1} > 0.$$

This requires that $v(s)$ shall have a zero on the $(m+1)$th interval and to the left of that of $u(s)$, so that $v(s)$ is negative near the right-hand end of the interval between the named consecutive zeros of $u(s)$, contrary to hypothesis. Hence $v(s)$ has a zero between these two consecutive zeros of $u(s)$.

If we take any solution $\bar{v}_i$ of (1.5) linearly independent of v_i, then the corresponding function $\bar{v}(s)$ has a zero between two consecutive zeros of $v(s)$ by theorem I. Hence $\bar{v}(s)$ has a zero on the interior of the interval between two consecutive zeros of $u(s)$. Combining this result with that in theorem I we have the following fundamental theorem of comparison:*

THEOREM II. *Let u_i and v_i be solutions of* (1.4) *and* (1.5), *respectively, and let $u(s)$ and $v(s)$ denote the functions into which they interpolate linearly with respect to a given system of coördinates. If $u(s)$ has consecutive zeros on the μth and $(m+1)$th intervals, $\mu < m$, then $v(s)$ has a zero between these zeros of $u(s)$ provided that either*

(*a*) *$\varphi_i \leqq \psi_i$, $i = \mu, \mu+1, \cdots, m$, the equality sign not holding for all these values; or,*

(*b*) *$\varphi_i = \psi_i$, $i = \mu, \mu+1, \cdots, m$, and the sets of constants u_i and v_i, for $i = \mu, \mu+1, \cdots, m$, are linearly independent.*

By means of this theorem we can readily establish other properties of comparison for $u(s)$ and $v(s)$. *Suppose that $u_1 \neq 0$, $v_1 \neq 0$, $\varphi_i \leqq \psi_i$ for $i = 1, 2, \cdots, \nu$, and that $u_2/u_1 > v_2/v_1$. Then, if $u(s)$ has k zeros on the first ν intervals of the coördinate system, $v(s)$ has at least k zeros on these intervals and the jth of these zeros* (*in increasing order*) *of $v(s)$ is to the left of the jth one of $u(s)$.* In view of theorem II it is sufficient to prove that the first zero of $v(s)$ is to the left of the first zero of $u(s)$. Let the $(m+1)$th interva be the one containing the first zero of $u(s)$. If $m = 0$ we employ the

* Compare the related theorem due to M. B. Porter, *Annals of Mathematics* (2), 3 (1902), p. 65.

relation $u_2/u_1 > v_2/v_1$ to show readily that $v(s)$ has a zero to the left of the first zero of $u(s)$. If $m > 0$ we employ (1.6) for $\mu = 1$. Either $v(s)$ vanishes to the left of the first zero of $u(s)$ or, on taking u_1 and v_1 positive (as we may do in proof without loss of generality), we have $u_{m+2}v_{m+1} - u_{m+1}v_{m+2} > 0$, so that $v(s)$ has a zero on the $(m + 1)$th interval to the left of that of $u(s)$ on this interval. This establishes the property in consideration.

Let u_1, v_1, u_{k+1}, v_{k+1} be all different from zero and let $u_2/u_1 > v_2/v_1$. Let $u(s)$ and $v(s)$ have the same number (which may be zero) of roots on the first k intervals. Then we have

$$\frac{u_{k+2}}{u_{k+1}} > \frac{v_{k+2}}{v_{k+1}}, \tag{1.7}$$

provided that $\varphi_i \leqq \psi_i$ for $i = 1, 2, \cdots, k$. From the result in the foregoing paragraph it follows that the rightmost root of $u(s)$ on these k intervals is to the right of that of $v(s)$, if either has a root on these intervals. Taking first the case in which they have at least one such root, let μ denote the number of the interval containing the rightmost root of $u(s)$ not at its right extremity. Without loss of generality in argument, we take $u_{\mu+1}$ and $v_{\mu+1}$ to be equal and positive. Then $v_\mu > u_\mu$ so that $u_{\mu+1}v_\mu > u_\mu v_{\mu+1}$. In (1.6) we take $m = k$. Then we have $u_{k+2}v_{k+1} > u_{k+1}v_{k+2}$, which reduces to (1.7). If $u(s)$ and $v(s)$ have no root on the k intervals, we take u_1 and v_1 positive (as we may in proof without loss of generality). We employ (1.6) with $\mu = 1$ and $m = k$. We have again $u_{k+2}v_{k+1} > u_{k+1}v_{k+2}$, whence (1.7) follows at once.

In like manner certain other similar results may be obtained from theorem II by modifying the form of certain of the inequalities in hypothesis and conclusion.

2. *Transcendental Oscillation Theorems.*—It is well known* that various types of transcendental equations may be realized as limiting cases of algebraic systems. For certain of these we shall now prove the propositions into which theorem I of § 1 passes under the appropriate limiting processes. For a homogeneous linear differential equation of the second order, $y'' + py' + qy = 0$, the result is classic in the Sturm theory: the zeros of any two linearly independent real solutions separate each other throughout any interval in which p and q are real-valued single-valued continuous functions of the real variable x.

Let us consider the difference equation

$$L(x)u(x) + M(x)u(x + 1) + N(x)u(x + 2) = 0 \tag{2.1}$$

in which all the indicated functions are real-valued single-valued continuous

* For references see a paper entitled "Algebraic Guides to Transcendental Problems" in *The Bulletin of the American Mathematical Society,* (2) 28 (1922): pp. 179–102.

functions of the real variable x for $x \geqq \alpha$, and $L(x)$ and $N(x)$ are both of one and the same sign for $x \geqq \alpha$. Let $u_1(x)$ and $u_2(x)$ be a fundamental system of solutions of this equation. Then, if we write

$$\omega(x) = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1(x+1) & u_2(x+1) \end{vmatrix}, \tag{2.2}$$

we have

$$\omega(x+1) = \begin{vmatrix} u_1(x+1) & u_2(x+1) \\ u_1(x+2) & u_2(x+2) \end{vmatrix} = \frac{L(x)}{N(x)}\,\omega(x),$$

the last member being gotten from the second last on replacing $u_i(x+2)$ $(i = 1, 2)$ by its value in terms of $u_i(x)$ and $u_i(x+1)$ gotten from the fact that $u_i(x)$ satisfies (2.1). It follows from this that the numbers $\omega(a)$, $\omega(a+1)$, $\omega(a+2)$, $\cdots$ are all of one sign (and hence not zero) if a is a real number not less than α and such that $\omega(a) \neq 0$.

Let us now consider the set of constants $u_i(a)$, $u_i(a+1)$, $u_i(a+2)$, $\cdots$ and let us interpolate them linearly into the function $\bar{u}_i(x)$ with respect to a system of coördinate axes obtained by drawing lines perpendicular to the x-axis through the points a, $a+1$, $a+2$, $\cdots$. Then from theorem I of § 1 and the remark following it we see that the zeros of the functions $\bar{u}_1(x)$ and $\bar{u}_2(x)$ separate each other throughout the range $a \leqq x < \infty$. Let the zeros of $\bar{u}_i(x)$ on the range $a \leqq x < \infty$ be called the *characteristic points* x for the function $u_i(x)$ with respect to the point a. Then a characteristic point is a point on a sub-interval $(a+k,\ a+k+1)$ of the coördinate system in which $u_i(x)$ has a zero, and in fact an odd number of zeros if it has no zero at an extremity of this interval, a zero being counted an odd or an even number of times according as the function does or does not change sign in the neighborhood of the zero. If $u_i(x)$ has a zero at one end of the interval $(a+k,\ a+k+1)$, it does not have a zero at the other end (since $\omega(a+k) \neq 0$) and the one zero at the end is in this case a characteristic point.

The results thus obtained may be put into the form of the following theorem:

THEOREM I. *Let $u_1(x)$ and $u_2(x)$ be a fundamental system of real-valued single-valued continuous solutions of equation* (2.1) *and let $\omega(x)$ be defined by* (2.2). *Let a be a real number not less than α for which $\omega(a) \neq 0$. Then the characteristic points x for the function $u_1(x)$ and those for the function $u_2(x)$, both with respect to a, separate each other; that is, between two consecutive characteristic points for one of these functions there is one and only one for the other function.**

* Compare the closely related theorem due to M. B. Porter, *Annals of Mathematics*, (2), 3 (1902), p. 65.

This result can be readily generalized to an extensive class of functional equations of the second order. Let us consider the substitution s,

$$x' = s_x,$$

and denote by $x' = s_x^n$ the nth power of this substitution. Let it be such that there exists an open interval I of the real axis of such sort that

$$\lim_{n=\infty} s_{x_0}^n = \beta$$

for every x_0 of I, β being an end-point of I and the limit being approached monotonically. [In the case of equation (2.1) we have $s_x \equiv x + 1$ while $\alpha < x < \infty$ is a suitable open interval I for every real value of α.] *Then consider the functional equation*

$$L(x)u(x) + M(x)u(s_x) + N(x)u(s_x^2) = 0 \tag{2.3}$$

in which the functions L, M, N are real-valued single-valued continuous functions of the real variable x on the interval I and $L(x)$ and $N(x)$ are both of one and the same sign on this interval. Suppose furthermore that s_x is such that equation (2.3) *has a fundamental system of solutions $u_1(x)$, $u_2(x)$ which are real-valued single-valued and continuous on I.*

Let a be a point of I and consider the set of constants $u_i(a)$, $u_i(s_a)$, $u_i(s_a^2)$, $\cdots$; let us interpolate them linearly into the function $\bar{u}_i(x)$ with respect to the system of coördinate axes obtained by drawing lines perpendicular to the x-axis through the points a, s_a, s_a^2, $\cdots$. Let the zeros of $\bar{u}_i(x)$ on the part of I which is between a and β, inclusive of a and exclusive of β, be called the *characteristic points* of $u_i(x)$ with respect to a. Then, by the same procedure as in the previous case, we have the following theorem:

THEOREM II. *Let $u_1(x)$, $u_2(x)$ be a fundamental system of real-valued single-valued continuous solutions of equation* (2.3) *and let a be a point of I for which $\omega(a) \neq 0$, where*

$$\omega(x) = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1(s_x) & u_2(s_x) \end{vmatrix}.$$

Then the characteristic points x for the function $u_1(x)$ and those for the function $u_2(x)$, both with respect to a, separate each other.

If we take $s_x \equiv qx$ where q is a real number greater [less] than unity, then a suitable interval I is that defined by the inequalities $\alpha < x < \infty$ [$\alpha > x > 0$] or that defined by the inequalities $-\alpha > x > -\infty$ [$-\alpha < x < 0$], where α is any positive number. We have therefore an interesting special case of the foregoing theorem applicable to q-difference equations of the form

$$L(x)u(x) + M(x)u(qx) + N(x)u(q^2x) = 0.$$

Each one of a great variety of functional equations finds a similar place here.

Let us next consider the analogous results for homogeneous linear

differential equations. They are classic for equations of the second order. Hence we may confine our present attention to equations of order n where $n > 2$. Such an equation we write in the general form

$$(2.4) \qquad u^{(n)} + p_1 u^{(n-1)} + \cdots + p_{n-1}u' + p_n u = 0,$$

where the superscripts refer to differentiation and where the coefficients p are real-valued single-valued continuous functions of the real variable x on the interval (ab). Since this involves an n-fold infinitude of solutions we shall require boundary conditions to restrict the permissible solutions to a two-fold infinitude so as to bring the present problem into the closest possible analogy with that involved in theorem I of § 1 and so that the new theorem shall indeed be a limiting case of that theorem. We shall suppose that these conditions are of the form

$$(2.5) \quad \sum_{j=1}^{\nu} \int_a^b L_{ij}(u)d\psi_{ij}(x) = 0, i = 1, 2, \cdots, n-2, \quad \nu = \text{positive integer},$$

the integrals being taken in the sense of Stieltjes, the functions $\psi_{ij}(x)$ being functions of bounded variation on (ab), and the $L_{ij}(u)$ denoting homogeneous linear differential expressions in u of order not greater than $n - 1$ (the case when some $L_{ij}(u)$ are of the form $g_{ij}u$ being included by convention as differential expressions of order zero). We assume that the conditions are so chosen that (2.4) has two and just two linearly independent solutions satisfying (2.5). This is the case, for instance, when the conditions reduce to the following: $u(a) = 0$, $u'(a) = 0$, $\cdots$, $u^{(n-3)}(a) = 0$.

Let $\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$ be a fundamental system of solutions of (2.4). Define the constants λ by means of the relations

$$(2.6) \qquad \sum_{k=1}^{\nu} \int_a^b L_{ik}(\bar{u}_j)d\psi_{ik}(x) = \lambda_{ij}, \qquad \begin{matrix} i = 1, 2, \cdots, n-2, \\ j = 1, 2, \cdots, n. \end{matrix}$$

Let u be a solution of (2.4) and write it in the form

$$(2.7) \qquad u = c_1\bar{u}_1 + c_2\bar{u}_2 + \cdots + c_n\bar{u}_n,$$

where $c_1, c_2, \cdots, c_n$ are constants. The conditions to be met in order that u shall also satisfy conditions (2.5) may now be written in the form

$$(2.8) \qquad \lambda_{i1}c_1 + \lambda_{i2}c_2 + \cdots + \lambda_{in}c_n = 0, \qquad i = 1, 2, \cdots, n-2.$$

The required condition that (2.4) and (2.5) shall have just two linearly independent simultaneous solutions now reduces to the condition that system (2.8), considered as a system for determining the coefficients $c_1, c_2, \cdots, c_n$, shall have just two linearly independent solutions; and for this it is a necessary and sufficient condition that the matrix $||\lambda_{ij}||$ of coefficients λ shall be of rank $n - 2$. Thus we have a necessary and sufficient condition

that the problem (2.4), (2.5) shall have two and just two linearly independent solutions.

If we write

$$u_1(x) = c_{11}\bar{u}_1 + c_{12}\bar{u}_2 + \cdots + c_{1n}\bar{u}_n,$$
$$u_2(x) = c_{21}\bar{u}_1 + c_{22}\bar{u}_2 + \cdots + c_{2n}\bar{u}_n,$$

where u_1 and u_2 are two linearly independent real solutions of (2.4), (2.5) and the c's are constants, then we have

$$\lambda_{i1}c_{11} + \lambda_{i2}c_{12} + \cdots + \lambda_{in}c_{1n} = 0, \qquad i = 1, 2, \cdots, n-2,$$
$$\lambda_{i1}c_{21} + \lambda_{i2}c_{22} + \cdots + \lambda_{in}c_{2n} = 0, \qquad i = 1, 2, \cdots, n-2.$$

It may be shown that a constant C exists, different from zero, such that

$$\begin{vmatrix} c_{1i} & c_{1j} \\ c_{2i} & c_{2j} \end{vmatrix} = CM_{ij},$$

where M_{ij} is the algebraic complement of

$$\begin{vmatrix} \bar{u}_i & \bar{u}_j \\ \bar{u}'_i & \bar{u}'_j \end{vmatrix}$$

in the expansion of the determinant

$$D(x) \equiv \begin{vmatrix} \bar{u}_1 & \bar{u}_2 & \cdots & \bar{u}_n \\ \bar{u}'_1 & \bar{u}'_2 & \cdots & \bar{u}'_n \\ \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \lambda_{n-2,\,1} & \lambda_{n-2,\,2} & \cdots & \lambda_{n-2,\,n} \end{vmatrix}$$

Now if we write

$$\omega(x) \equiv \begin{vmatrix} u_1(x) & u_2(x) \\ u'_1(x) & u'_2(x) \end{vmatrix}$$

we have

$$\omega(x) \equiv \begin{vmatrix} c_{11}\bar{u}_1 + \cdots + c_{1n}\bar{u}_n & c_{21}\bar{u}_1 + \cdots + c_{2n}\bar{u}_n \\ c_{11}\bar{u}'_1 + \cdots + c_{1n}\bar{u}'_n & c_{21}\bar{u}'_1 + \cdots + c_{2n}\bar{u}'_n \end{vmatrix}$$
$$\equiv \sum_{\substack{i,j=1 \\ i<j}}^{n} \begin{vmatrix} c_{1i} & c_{1j} \\ c_{2i} & c_{2j} \end{vmatrix} \cdot \begin{vmatrix} \bar{u}_i & \bar{u}_j \\ \bar{u}'_i & \bar{u}'_j \end{vmatrix}$$
$$\equiv \sum_{\substack{i,j=1 \\ i<j}}^{n} CM_{ij} \begin{vmatrix} \bar{u}_i & \bar{u}_j \\ \bar{u}'_i & \bar{u}'_j \end{vmatrix} \equiv CD(x).$$

From this it follows that the zeros of $\omega(x)$ on (ab) are the same as those of $D(x)$ on (ab).*

* This result can be generalized to the case in which the range of i in (2·5) is over the set 1, 2, $\cdots$, k, where k is any one of the numbers 1, 2, $\cdots$, $n-1$, these conditions being so chosen that the number of linearly independent solutions of (2·4) subject to the new conditions is $n-k$. If such linearly independent solutions are denoted by u_1, u_2, $\cdots$, u_{n-k} and if constants λ are defined as before by means of a fundamental system of solutions $\bar{u}_1$, $\bar{u}_2$, $\cdots$, $\bar{u}_n$ of (2·4), then for a suitable non-zero constant C we have the identity

Let us consider any second fundamental system of solutions of (2.4). The functions in it can be represented in the form

$$\gamma_{i1}\bar{u}_1 + \gamma_{i2}\bar{u}_2 + \cdots + \gamma_{in}\bar{u}_n, \qquad i = 1, 2, \cdots, n,$$

where the γ's are constants such that the determinant $|\gamma_{ij}|$ is different from zero. If we form the determinant $D_1(x)$ of this new system of solutions, corresponding to $D(x)$ for the former system, it is easy to show by means of the multiplication theorem for determinants that we have $D_1(x) = |\gamma_{ij}|D(x)$. Hence the zeros of $D(x)$ in (ab) are independent of the fundamental system of solutions employed. The points a, b and these zeros therefore define a division of the interval (ab) into sub-intervals $I_1, I_2, I_3, \cdots$ such that $D(x)$ vanishes at the extremities of these segments other than the points a and b and does not vanish in the interior of any of these segments; and this division of (ab) into sub-intervals depends solely on the equation (2.4) and the boundary conditions (2.5).

We may now readily prove the following theorem:

THEOREM III. *If u_1 and u_2 are any two linearly independent real solutions of* (2.4), (2.5), *then between any two consecutive zeros of one of these solutions in the interior of one of the intervals $I_1, I_2, I_3, \cdots$ lies one and only one zero of the other solution.*

Since $\omega(x) \equiv CD(x)$ the determinant $\omega(x)$ is of one sign in the interior of an interval I_k. Without loss of generality we may take it to be positive in the interior of I_k; doing this we have

$$u_1u_2' - u_1'u_2 > 0 \tag{2.9}$$

within I_k. Let α and β ($\alpha < \beta$) be two consecutive zeros of u_1 on I_k. We may (and we will) take u_1 to be positive in the interior of the interval $(\alpha\beta)$, since if it were negative we could replace u_1 and u_2 by $-u_1$ and $-u_2$ without disturbing any other assumed properties or relations. Then $u_1'(\alpha) > 0$ and $u_1'(\beta) < 0$. Then, since $u_1(\alpha) = 0 = u_1(\beta)$, it follows from (2.9) that $u_2(\alpha) < 0$ and $u_2(\beta) > 0$. Hence there is one zero of $u_2(x)$ between two consecutive zeros of $u_1(x)$ in the interior of I_k. There cannot be more than one, since the result just obtained can be used to show that a zero of $u_1(x)$

$$\begin{vmatrix} u_1 & u_2 & \cdots & u_{n-k} \\ u_1' & u_2' & \cdots & u_{n-k}' \\ \cdot & \cdot & \cdot & \cdot \\ u_1^{(n-k-1)} & u_2^{(n-k-1)} & \cdots & u_{n-k}^{(n-k-1)} \end{vmatrix} \equiv C \begin{vmatrix} \bar{u}_1 & \bar{u}_2 & \cdots & \bar{u}_n \\ \bar{u}_1' & \bar{u}_2' & \cdots & \bar{u}_n' \\ \cdot & \cdot & \cdot & \cdot \\ \bar{u}_1^{(n-k-1)} & \bar{u}_2^{(n-k-1)} & \cdots & \bar{u}_n^{(n-k-1)} \\ \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \lambda_{k1} & \lambda_{k2} & \cdots & \lambda_{kn} \end{vmatrix}.$$

It is possible to use this result to obtain certain generalizations of theorem III and of the consequences which result from it; but when $k < n - 2$ the results do not maintain an elegance comparable with that when $k = n - 2$.

lies between any two consecutive zeros of $u_2(x)$ on I_k. Hence the theorem is established.

Let us now consider the problem of applying the result in the foregoing theorem to the case of any two linearly independent solutions of (2.4) without reference to any preassigned boundary conditions. Let u_1 and u_2 denote any two linearly independent solutions of (2.4). Holding these solutions fixed let us consider them and the differential equation in connection with certain *associated boundary conditions* of the form (2.5). By conditions *associated* with the given differential equation and given solutions we shall mean any $n - 2$ conditions capable of expression in the form (2.5) and such that the solutions of (2.4) and the determined conditions (2.5) are those functions and those alone which are linearly dependent upon u_1 and u_2.

As a simple example of such associated boundary conditions we have those determined as follows: Let the initial constants for u_1 and u_2 at a point $x = \alpha$ of (ab) be

$$u_i^{(k)}(\alpha) = \rho_{ik}, \qquad k = 0, 1, 2, \cdots, n - 1, \qquad i = 1, 2.$$

Let the coefficients σ_{ij} be so chosen that the equations

$$(2.10) \quad \sigma_{i0}u(\alpha) + \sigma_{i1}u'(\alpha) + \cdots + \sigma_{i,\,n-1}u^{(n-1)}(\alpha) = 0, \qquad i = 1, 2, \cdots, n - 2$$

have those solutions and those alone which may be written in the form

$$u^{(k)}(\alpha) = a_1\rho_{1k} + a_2\rho_{2k}, \qquad k = 0, 1, \cdots, n - 1,$$

where a_1 and a_2 are arbitrary constants. Then the solutions of (2.4) and (2.10) are those functions and those alone which are linearly dependent upon u_1 and u_2. Such conditions (2.10) afford a special form of boundary conditions to be associated with the equation (2.4) and the given linearly independent solutions u_1 and u_2.

By means of the so determined boundary conditions (2.5) and a fundamental system $\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$ of solutions of (2.4) we may define a determinant $D(x)$, as in the earlier treatment when the conditions (2.5) were preassigned. Let the zeros of $D(x)$ on (ab) be called the *special points** of (ab) with respect to the given solutions u_1 and u_2 and the named associated boundary conditions. These special points are independent of the particular fundamental system of solutions used in defining them. By aid of the foregoing theorem we now have readily the following theorem:

THEOREM IV. *If u_1 and u_2 are any two linearly independent solutions of* (2.4), *then between any two consecutive zeros α, β of u_1 on (ab) there is one*

* It is clear that the special points are not altered if u_1 and u_2 are replaced by any two linearly independent functions which are linearly dependent upon them.

and just one zero of u_2 *or there is on the interval* $(\alpha\beta)$ *a special point of* (ab) *with respect to the given solutions* u_1 *and* u_2 *and any set of boundary conditions associated with* (2.4) *and the given solutions* u_1 *and* u_2 *in the way indicated.*

As a very simple example illustrating the first of the two foregoing theorems let us consider the equation and condition

$$u'''(x) + u'(x) = 0, \qquad u''(0) + u(0) = 0.$$

A fundamental system of solutions of the differential equation is 1, sin x, cos x. The determinant $D(x)$ formed with these is now

$$D(x) \equiv \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 1 & 0 & 0 \end{vmatrix} \equiv 1.$$

Hence there are no special points. Hence any two linearly independent solutions of the given differential equation satisfying the given boundary conditions have together the property that their zeros separate each other.

In illustration of the last foregoing theorem, let us consider the two solutions

$$u_1 = a + \sin x, \qquad u_2 = \cos x$$

of the same differential equation $u'''(x) + u'(x) = 0$, where a is a given real constant. We have

$$u_1(0) = a,\ \ u_1'(0) = 1,\ \ u_1''(0) = 0;\ \ u_2(0) = 1,\ \ u_2'(0) = 0,\ \ u_2''(0) = -1.$$

As a single associated boundary condition relevant in this case we may take

$$u(0) - au'(0) + u''(0) = 0.$$

Then with the fundamental system 1, sin x, cos x of solutions we find that $D(x)$ has the value

$$D(x) \equiv \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 1 & -a & 0 \end{vmatrix} \equiv -(a \sin x + 1).$$

Then the corresponding special points are the zeros of $a \sin x + 1$. If $|a| < 1$ they are absent and the zeros of u_1 and u_2 separate each other. If $|a| \geqq 1$ the special points interfere with this characteristic relative distribution of the zeros of u_1 and u_2. If $|a| > 1$ it is clear that u_1 has no zeros so that in this case at least one special point must occur on every interval containing two zeros of u_2.

We have assumed that the differential equation (2.4) is of order greater than 2. It is clear that a similar and in fact much simpler analysis of the same general sort as that given above is applicable to the case when $n = 2$

and that the principal theorem which thus results is the classic theorem of Sturm. The two foregoing theorems may be considered as the natural extension of this in one important direction. Some elements of elegance are absent from the more general results, a fact which is inevitable as the simplest considerations show and as is implicitly apparent from the examples just given. The present general results afford fairly definite information as to the way in which the situation is complicated when $n > 2$.

Let $s_n(x)$ *denote either* $x + n$ *or* $q^n x$, *where* q *is a real number greater than unity,** *and let us consider the functional equation of order* n *greater than* 2,

$$(2.11)\quad u\{s_n(x)\} + p_1 u\{s_{n-1}(x)\} + \cdots + p_{n-1}u\{s_1(x)\} + p_n u\{x\} = 0,$$

in which the coefficients $p_1, p_2, \cdots, p_n$ *are real-valued single-valued continuous functions of the real variable* x *for* $x \geqq \alpha$ (α *being positive for the case of the* q*-difference equation*). Let $u_1(x)$ and $u_2(x)$ be two real-valued single-valued continuous solutions of (2.11) which are linearly independent with respect to periodic multipliers $P(x)$ such that $P\{s_1(x)\} \equiv P(x)$. Let a be a point such that $a \geqq \alpha$ and $\omega(a) \neq 0$, where

$$\omega(x) \equiv \begin{vmatrix} u_1\{x\} & u_2(x) \\ u_1\{s_1(x)\} & u_2\{s_1(x)\} \end{vmatrix}.$$

In connection with these solutions we consider *associated boundary conditions* which are capable of representation by means of Stieltjes integrals in the form

$$(2.12)\qquad \int_a^b u\,d\psi_i(x) = 0, \qquad i = 1, 2, \cdots, n-2,$$

where b is any conveniently chosen number greater than a and the $\psi_i(x)$ are functions of bounded variation in (ab), these boundary conditions being selected in such a way that the problem (2.11), (2.12) has for real-valued single-valued continuous solutions those functions and those alone for which it is true that

$$(2.13)\quad u\{s_k(a)\} = c_1 u_1\{s_k(a)\} + c_2 u_2\{s_k(a)\}, \qquad k = 0, 1, 2, \cdots,$$

c_1 and c_2 being arbitrary constants. A particular set of such boundary conditions may be written in the form

$$\sigma_{i0}u\{a\} + \sigma_{i1}u\{s_1(a)\} + \cdots + \sigma_{i,\,n-1}u\{s_{n-1}(a)\} = 0, \qquad i = 1, 2, \cdots, n-2,$$

* The results in the remainder of this section can be extended without difficulty to a class of functional equations obtained from those treated here by replacing $s_n(x)$ by s_x^n where $x' = s_x^n$ is the nth power of the substitution $x' = s_x$ treated in connection with the second theorem of this section. Moreover, in the case of a q-difference equation in which q is positive and less than unity we may employ the transformation $x = 1/t$, $u(x) = v(t)$ and so obtain a new q-difference equation included in the class treated in the text. Similarly, on replacing x by $-t$ one changes from a q-difference equation with negative q to one with positive q.

where the σ_{ij} are so chosen that this algebraic system has those solutions and those alone which may be written in the form of the first n equations (2.13), c_1 and c_2 being arbitrary constants.

Let $\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$ be a fundamental system of solutions of (2.11) and let constants λ be defined by means of the relations

$$\int_a^b \bar{u}_j d\psi_i(x) = \lambda_{ij}, \qquad \begin{array}{l} i = 1, 2, \cdots, n-2, \\ j = 1, 2, \cdots, n. \end{array} \tag{2.14}$$

Then if we write

$$D(x) \equiv \begin{vmatrix} \bar{u}\{x\} & \bar{u}_2\{x\} & \cdots & \bar{u}_n(x) \\ \bar{u}_1\{s_1(x)\} & \bar{u}_2\{s_1(x)\} & \cdots & \bar{u}_n\{s_1(x)\} \\ \lambda_{11} & \lambda_{12} & \cdots & \lambda_{1n} \\ \lambda_{21} & \lambda_{22} & \cdots & \lambda_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ \lambda_{n-2,\,1} & \lambda_{n-2,\,2} & \cdots & \lambda_{n-2,\,n} \end{vmatrix}$$

we may readily show that $\omega\{s_k(a)\} = CD\{s_k(a)\}$, where C is independent of k, the method of proof being similar to that by which the relation $\omega(x) \equiv CD(x)$ was proved for the foregoing problem in differential equations.

By a *characteristic point* of any function $v(x)$ with respect to a we shall mean a zero (to the right of or at a) of the function $\bar{v}(x)$ obtained by linear interpolation from the constants $v\{a\}$, $v\{s_1(a)\}$, $v\{s_2(a)\}$, $\cdots$ with respect to a set of axes obtained by drawing perpendiculars to the x-axis through the points a, $s_1(a)$, $s_2(a)$, $\cdots$. The characteristic points of $D(x)$ with respect to a we shall call the *special points* for the solutions $u_1(x)$ and $u_2(x)$ with respect to the boundary conditions (2.12). They are clearly independent of the choice of fundamental system $\bar{u}_1, \bar{u}_2, \cdots, \bar{u}_n$ used in defining them.

The quantities $\omega\{s_k(a)\}$ are all of one sign if the variation of k is so restricted that all the points $s_k(a)$ lie on a single interval containing no special points. Then through use of the method employed in proving the first two theorems in this section we are led easily to the following theorem:

THEOREM V. *If u_1 and u_2 are any two real-valued single-valued continuous solutions of* (2.11) *which are independent, then between any two consecutive characteristic points α, β for u_1 with respect to a point a for which $\omega(a) \neq 0$ there is one and just one characteristic point for u_2 or there is on the interval $(\alpha\beta)$ a special point with respect to the given solutions u_1 and u_2 and any set of boundary conditions associated with* (2.11) *and the given solutions u_1 and u_2 in the way indicated.*

By taking $s_k(x) \equiv x + k$, we have in this theorem the extension, to equations of order greater than two, of the results stated for a second order difference equation in the first theorem of this section; and by taking

$s_k(x) \equiv q^k x,\ q > 1$, we have a corresponding theorem for q-difference equations.

3. *Transcendental Comparison Theorems.*—The results stated in theorem II of § 1 and the following paragraphs have as limiting cases certain of the classic theorems of comparison of Sturm for homogeneous linear differential equations of the second order; and they were indeed suggested by these Sturmian theorems. The latter, with considerable loss of elegance, have been extended to homogeneous linear differential equations of general order k.* It is possible to extend the algebraic results of the latter part of § 1 to the analogous algebraic case, namely, the case of algebraic systems with k linearly independent solutions; but the results lack (in some respects) the desired elegance. From them one may in turn obtain corresponding properties of a certain class of functional equations. We content ourselves with giving these results for the most interesting case, namely, that in which the equations have just two independent solutions.

Let us consider the functional equations

$$u(s_x^2) + \varphi(x)u(s_x) + u(x) = 0, \tag{3.1}$$

$$v(s_x^2) + \psi(x)v(s_x) + v(x) = 0, \tag{3.2}$$

in which $\varphi(x)$ and $\psi(x)$ are real-valued single-valued continuous functions of the real variable x on the interior of the interval I, where s_x, s_x^2, I are defined as in the paragraph following theorem I of § 2. If a is an interior point of the interval I, we define the *characteristic points* of a function $t(x)$ with respect to a to be the zeros on the interval $a \leqq x < \beta$ (or $a \geqq x > \beta$) of the function $t(x)$ derived from the constants $t(a)$, $t(s_a)$, $t(s_a^2)$, $\cdots$ by linear interpolation with respect to the system of coördinate axes obtained by drawing lines perpendicular to the x-axis through the points a, s_a, s_a^2, $\cdots$.

From theorem II of § 1 we have at once the following theorem:

THEOREM I. *Let u and v be real-valued single-valued continuous solutions of equations* (3.1) *and* (3.2), *respectively. If $u(x)$ has two consecutive characteristic points with respect to a on the μth and $(m + 1)$th intervals $(\mu < m)$ of the set of intervals whose end-points are the consecutive pairs of the sequence a, s_a, s_a^2, $\cdots$, then $v(x)$ has a characteristic point between these two characteristic points of u provided that either*

(*a*) *$\varphi(x) \leqq \psi(x)$ at the end-points of each of these intervals from the μth to the mth inclusive, the equality sign not holding for all these end-points; or,*

(*b*) *$\varphi(x) = \psi(x)$ at the end-points of each of these intervals from the μth to the mth inclusive and the two sets of constants,*

$$u(s_a^{\mu-1}),\ u(s_a^{\mu}),\ \cdots,\ u(s_a^{m-1}); \qquad v(s_a^{\mu-1}),\ v(s_a^{\mu}),\ \cdots,\ v(s_a^{m-1}),$$

are linearly independent.

* *Annals of Mathematics* (2), 19 (1918): 159–171.

This theorem affords certain properties of comparison for the relative distribution of the zeros of $u(x)$ and $v(x)$. Other related properties may be obtained similarly from the italicized results following theorem II of § 1; we state them without further elaboration of the proof.

THEOREM II. *Let u and v be real-valued single-valued continuous solutions of equations* (3.1) *and* (3.2), *respectively. Let us suppose that* $u(a) \neq 0$, $v(a) \neq 0$, $u(s_a)/u(a) > v(s_a)/v(a)$; *and that* $\varphi(x) \leqq \psi(x)$ *for* $x = a, s_a, s_a^2, \cdots, s_a^{\nu-1}$. *If $u(x)$ has k characteristic points on the ν intervals whose end-points are the consecutive pairs of the sequence* $a, s_a, s_a, \cdots, s_a^\nu$, *then $v(x)$ has at least k characteristic points on these intervals; and the jth of these characteristic points (counted from a towards s_a^ν) of $v(x)$ is nearer to a than the jth characteristic point of $u(x)$.*

THEOREM III. *Let u and v be real-valued single-valued continuous solutions of equations* (3.1) *and* (3.2), *respectively. Let* $u(a)$, $v(a)$, $u(s_a^k)$, $v(s_a^k)$ *be all different from zero and let* $u(s_a)/u(a) > v(s_a)/v(a)$. *Let $u(x)$ and $v(x)$ have the same number (which may be zero) of characteristic points on the k intervals whose end-points are the consecutive pairs of the sequence* $a, s_a, s_a^2, \cdots, s_a^k$. *Then we have*

$$\frac{u(s_a^{k+1})}{u(s_a^k)} > \frac{v(s_a^{k+1})}{v(s_a^k)},$$

provided that $\varphi(x) \leqq \psi(x)$ *for* $x = a, s_a, \cdots, s_a^{k-1}$.

Since the point a (in each of these theorems) has a great range of variation, the stated results give a large measure of information about the relative distribution of the zeros of $u(x)$ and $v(x)$ even though the arbitrary elements of the solutions of (3.1) and (3.2) are functions (rather than constants) restricted merely by a relation of the form $p(s_x) = p(x)$ and certain considerations of reality and continuity. They have their greatest use in yielding information concerning the solutions of a given equation (3.2) by comparing them with some simple functions which are known to be solutions of equation (3.1) with appropriate determination of the $\varphi(x)$.

4. *A Generalization of Boundary Conditions for Expansion Problems.*—By means of the differential expressions

$$L(u) \equiv l_n \frac{d^n u}{dx^n} + l_{n-1} \frac{d^{n-1} u}{dx^{n-1}} + \cdots + l_1 \frac{du}{dx} + l_0 u,$$

$$L_1(u) \equiv g_m \frac{d^m u}{dx^m} + \cdots + g_1 \frac{du}{dx} + g_0 u, \qquad m < n,$$

in which the coefficients l_k, g_k with their derivatives of all orders are real-valued single-valued continuous functions of x in an interval (ab) and l_n is positive throughout (ab) while g_m does not vanish in (ab) [and hence is

either positive or negative throughout (ab)], we define the differential equation

$$L(u) + \lambda L_1(u) = 0 \tag{4.1}$$

and its adjoint

$$M(v) + \lambda M_1(v) = 0, \tag{4.2}$$

where $M(v)$ and $M_1(v)$ are the adjoints of $L(u)$ and $L_1(u)$, respectively. We have classic identities of the form

$$vL(u) - uM(v) \equiv \frac{d}{dx}\{R(u, v)\}, \quad vL_1(u) - uM_1(v) \equiv \frac{d}{dx}\{R_1(u, v)\}. \tag{4.3}$$

The quantity

$$[R(u, v) + \lambda R_1(u, v)]_{x=a}^{x=b}$$

is a bilinear form in the two sets of variables

$$u^{(n-1)}(a),\ u^{(n-2)}(a),\ \cdots,\ u'(a),\ u(a),\ u^{(n-1)}(b),\ \cdots,\ u'(b),\ u(b)$$

and

$$v(a),\ v'(a),\ \cdots,\ v^{(n-2)}(a),\ v^{(n-1)}(a),\ v(b),\ v'(b),\ \cdots,\ v^{(n-1)}(b).$$

If this form is arranged in a square array whose columns contain the u's in the order written and whose rows contain the v's in the order written, then the matrix has the following properties: every element below the main diagonal is zero; every element in the upper right-hand fourth of the matrix is zero, the division being made by horizontal and vertical lines; the determinant of the matrix has the positive value $\{l_n(a)l_n(b)\}^n$.

This bilinear expression may be written in an infinite number of ways in the form

$$[R(u, v) + \lambda R_1(u, v)]_{x=a}^{x=b} \equiv \sum_{i=1}^{2n} \{U_{1i}(u) + \lambda U_{2i}(u)\}\{V_{1i}(v) + \lambda V_{2i}(v)\}, \tag{4.4}$$

where the U_{1i}, U_{2i}, V_{1i}, V_{2i} are linear homogeneous functions with constant coefficients, the first two in the variables u and the last two in the variables v, such that the determinant of the linear forms $U_{1i}(u) + \lambda U_{2i}(u)$ in the variables u is independent of λ and different from zero and such that the determinant of the linear forms $V_{1i}(v) + \lambda V_{2i}(v)$ in the variables v has the same properties. Then with (4.1) we associate the boundary conditions

$$U_{1i}(u) + \lambda U_{2i}(u) = 0, \qquad i = 1, 2, \cdots, n; \tag{4.5}$$

and with the adjoint equation (4.2) the *adjoint* boundary conditions

$$V_{1i}(v) + \lambda V_{2i}(v) = 0, \qquad i = n + 1, \cdots, 2n. \tag{4.6}$$

We say that the problem (4.1), (4.5) and the problem (4.2), (4.6) are each the *adjoint* of the other.*

The following theorems may now be proved in the same way as the special cases of them are proved in the article just cited:

THEOREM I. *If for* $\lambda = \bar{\lambda}$ *a solution* $\bar{u}(x)$ (*not identically zero*) *of* (4.1), (4.5) *exists, then a solution* $\bar{v}(x)$ *of* (4.2), (4.6) *also exists for* $\lambda = \bar{\lambda}$; *if* $\bar{u}(x)$ *is unique* (*except for a constant factor*), $\bar{v}(x)$ *is also unique* (*except for a constant factor*).

THEOREM II. *If* $y_1, y_2, \cdots, y_n$ *are n linearly independent solutions of* (4.1) *for* $\lambda = \bar{\lambda}$, *the condition that* $\bar{\lambda}$ *is a characteristic value is that the determinant*

$$\Delta = \begin{vmatrix} U_{11}(y_1) + \bar{\lambda}U_{21}(y_1) & U_{11}(y_2) + \bar{\lambda}U_{21}(y_2) & \cdots & U_{11}(y_n) + \bar{\lambda}U_{21}(y_n) \\ U_{12}(y_1) + \bar{\lambda}U_{22}(y_1) & U_{12}(y_2) + \bar{\lambda}U_{22}(y_2) & \cdots & U_{12}(y_n) + \bar{\lambda}U_{22}(y_n) \\ \cdot & \cdot & \cdot & \cdot \\ U_{1n}(y_1) + \bar{\lambda}U_{2n}(y_1) & U_{1n}(y_2) + \bar{\lambda}U_{2n}(y_2) & \cdots & U_{1n}(y_n) + \bar{\lambda}U_{2n}(y_n) \end{vmatrix}$$

shall vanish; a characteristic value $\bar{\lambda}$ *of* λ *is simple when and only when some first minor of* Δ *does not vanish.*

Here the terms *characteristic value* and *simple characteristic value* are used in the same sense as in the treatment of the special case referred to.

From the equations

$$L(u_i) + \lambda_i L_1(u_i) = 0, \qquad M(v_j) + \lambda_j M_1(v_j) = 0$$

we have

$$\{v_j[L(u_i) + \lambda_i L_1(u_i)] - u_i[M(v_j) + \lambda_i M_1(v_j)]\} + (\lambda_i - \lambda_j)u_i M_1(v_j) = 0.$$

Integrating with respect to x from a to b and simplifying by use of equations (4.3) and (4.4) and the boundary conditions

$$\begin{aligned} U_{1t}(u_i) + \lambda_i U_{2t}(u_i) &= 0, \qquad t = 1, 2, \cdots, n, \\ V_{1t}(v_j) + \lambda_i V_{2t}(v_j) - (\lambda_i - \lambda_j)V_{2t}(v_j) &= 0, \qquad t = n+1, \cdots, 2n, \end{aligned}$$

we have

$$(\lambda_i - \lambda_j)\sum_{k=n+1}^{2n} V_{2k}(v_j)\{U_{1k}(u_i) + \lambda_i U_{2k}(u_i)\} + (\lambda_i - \lambda_j)\int_a^b u_i M_1(v_j)dx = 0.$$

From this relation and that obtained by interchanging the rôles of u and v we deduce at once the following theorem, generalizing theorem III of the preceding paper:

THEOREM III. *If* $u_i(x)$ *and* $v_j(x)$ *are solutions of* (4.1), (4.5) *and* (4.2),

* For a special case of these adjoint problems, with references to the literature, see AMERICAN JOURNAL OF MATHEMATICS, 43 (1921): 232–270.

(4.6), *respectively, the first for* $\lambda = \lambda_i$ *and the second for* $\lambda = \lambda_j$ *and if* $\lambda_i \neq \lambda_j$, *then we have*

$$\int_a^b u_i M_1(v_j)dx + \sum_{k=n+1}^{2n} V_{2k}(v_j)\{U_{1k}(u_i) + \lambda_i U_{2k}(u_i)\} = 0,$$

$$\int_a^b v_j L_1(u_i)dx + \sum_{k=1}^{n} U_{2k}(u_i)\{V_{1k}(v_j) + \lambda_j V_{2k}(v_j)\} = 0.$$

The novelty in this form of the problem is in the appearance of the parameter λ in the boundary conditions. It is clear that a corresponding generalization exists for the various boundary value and expansion problems treated in the memoir mentioned above; and that the development of the theory follows lines closely parallel to the earlier treatment.

5. *Expansion Problems for q-Difference and Integro-q-Difference Equations.*—Let us consider the adjoint systems of q-difference equations

$$u_i(qx) - u_i(x) = \sum_{j=1}^{n} (\varphi_{ij} + \lambda\psi_{ij})u_j(x), \qquad i = 1, 2, \cdots, n, \tag{5.1}$$

$$v_i(x) - v_i(qx) = \sum_{j=1}^{n} (\varphi_{ji} + \lambda\psi_{ji})v_j(qx), \qquad i = 1, 2, \cdots, n, \tag{5.2}$$

where q is a constant whose absolute value is different from unity and where the φ_{ij} and ψ_{ij} are functions of x which are analytic at infinity and have a zero there. These systems of equations possess fundamental systems*

$$u_{1j}(x),\ u_{2j}(x),\ \cdots,\ u_{nj}(x); \qquad v_{1j}(x),\ v_{2j}(x),\ \cdots,\ v_{nj}(x)$$

of solutions each function of which is analytic at infinity, say, analytic for $|x| \geqq R$, R being an appropriately chosen positive constant; moreover, the constant term in the solution $u_{ij}(x)$, and that in the solution $v_{ij}(x)$ as well, is δ_{ij} where δ_{ij} denotes unity or zero according as j is or is not equal to i. Any solution which is analytic at infinity is made up from the foregoing solutions by taking linear combinations of them with constant coefficients. We confine attention to such solutions of (5.1) and (5.2) as are analytic for $|x| \geqq R$.

If we multiply (5.1) member by member by $v_i(qx)$ and (5.2) by $-u_i(x)$, add the resulting equations member by member, and sum as to i from 1 to n, we have

$$\sum_{i=1}^{n} \delta\{u_i(x)v_i(x)\} = 0, \tag{5.3}$$

where δ denotes the operation defined by the relation $\delta f(x) \equiv f(qx) - f(x)$. Let a be a number such that $|a| \geqq R$. In (5.3) sum as to x from a to ∞,

* These existence theorems are readily proved by means of the theory of power series. There also exists a like theory in which the point zero plays the rôle played here by the point infinity.

where x runs over the values $a, qa, q^2a, \cdots$ or the values $a, q^{-1}a, q^{-2}a, \cdots$ according as $|q|$ is greater than or less than unity; thus we have

$$(5.4) \qquad \sum_{i=1}^{n} \{u_i(\infty)v_i(\infty) - u_i(a)v_i(a)\} = 0.$$

The first member of this equation is the analogue, for the present theory, of the bilinear form in the first member of equation (4.4). It is a bilinear form in the two sets of $2n$ variables each

$$u_1(\infty),\ u_2(\infty),\ \cdots,\ u_n(\infty),\ u_1(a),\ \cdots,\ u_n(a);$$
$$v_1(\infty),\ v_2(\infty),\ \cdots,\ v_n(\infty),\ v_1(a),\ \cdots,\ v_n(a).$$

Since the bilinear form is obviously of rank $2n$ it may be written in an infinite number of ways in the normal form

$$(5.5) \qquad \sum_{i=1}^{n} \{u_i(\infty)v_i(\infty) - u_i(a)v_i(a)\} \equiv \sum_{i=1}^{2n} U_i(u)V_i(v),$$

where the $U_i(u)[V_i(v)]$ are homogeneous linear expressions (with constant coefficients) in the variables $u[v]$. Then with our systems of q-difference equations we associate the boundary conditions

$$(5.6) \qquad U_i(u) = 0, \qquad i = 1, 2, \cdots, n;$$
$$(5.7) \qquad V_i(v) = 0, \qquad i = n+1, \cdots, 2n.$$

Then relation (5.4) is satisfied in virtue of the boundary conditions alone.

THEOREM I. *If for* $\lambda = \bar{\lambda}$ *a solution* $\bar{u}_i(x)$ (*not identically zero*) *of* (5.1), (5.6) *exists, then a solution* $\bar{v}_i(x)$ *of* (5.2), (5.7) *also exists for* $\lambda = \bar{\lambda}$; *if the solution* $\bar{u}_i(x)$ *is unique* (*except for a constant factor*), *the solution* $\bar{v}_i(x)$ *is also unique* (*except for a constant factor*).

For $\lambda = \bar{\lambda}$ we have

$$U_1(\bar{u}) = U_2(\bar{u}) = \cdots = U_n(\bar{u}) = 0, \qquad U_{n+k}(\bar{u}) \neq 0,$$

the inequality holding for some k of the set $1, 2, \cdots, n$, since from the relations $U_i(\bar{u}) = 0$ for $i = 1, 2, \cdots, 2n$ it would follow that $\bar{u}_i(\infty) = 0$ for $i = 1, 2, \cdots, n$, so that $\bar{u}_i$ would be identically zero, contrary to hypothesis. In the n-fold totality of solutions of (5.2) for $\lambda = \bar{\lambda}$, there is at least one, say $\bar{v}_i(x)$, which satisfies the $n - 1$ conditions

$$V_{n+j}(\bar{v}) = 0,$$

where j runs over all the numbers of the set $1, 2, \cdots, n$ except k. But (5.4) must be satisfied when u_i and v_i are replaced by $\bar{u}_i$ and $\bar{v}_i$, respectively, since the latter are taken for the same value of λ. Thence, through aid of the identity (5.5) and the boundary conditions for $\bar{u}_i$ and $\bar{v}_i$ already verified, we have $U_{n+k}(\bar{u})V_{n+k}(\bar{v}) = 0$. Therefore $V_{n+k}(\bar{v}) = 0$, so that the solution $\bar{v}_i(x)$ satisfies all the boundary conditions (5.7).

It remains to show that $\bar{v}_i(x)$ is unique whenever $\bar{u}_i(x)$ is unique. We shall prove this by supposing that the solution $\bar{v}_i(x)$ is not unique and then showing that the solution $\bar{u}_i(x)$ can not be unique. Since $\bar{v}_i(x)$ is now supposed to be not unique, let $\bar{v}_i(x)$ and $v_i^*(x)$ be two linearly independent solutions of (5.2) and (5.7) for $\lambda = \bar{\lambda}$. Then different numbers j and k of the set 1, 2, $\cdots$, n exist such that

$$V_j(\bar{v}) \neq 0, \qquad V_k(v^*) \neq 0.$$

If then

$$V_j(\bar{v})V_k(v^*) = V_j(v^*)V_k(\bar{v}),$$

we have

$$V_j(v^*) \neq 0, \qquad V_k(\bar{v}) \neq 0,$$

so that the functions

$$v_i(x) = V_j(v^*)\bar{v}_i(x) - V_j(\bar{v})v^*(x)$$

afford a solution of (5.2), (5.7) which is not identically zero. If

$$(5.8) \qquad \begin{vmatrix} V_j(\bar{v}) & V_\alpha(\bar{v}) \\ V_j(v^*) & V_\alpha(v^*) \end{vmatrix}$$

is zero for every α of the set 1, 2, $\cdots$, n, then we shall have $V_i(v) = 0$ for $i = 1, 2, \cdots, 2n$, a result which is impossible since it would imply that $v_i(\infty) = 0$ $(i = 1, 2, \cdots, n)$ and hence that $v_i(x)$ is the identically zero solution (contrary to the hypothesis about the linear independence of $\bar{v}_i(x)$ and $v_i^*(x)$). Hence for some α of the set 1, 2, $\cdots$, n the determinant (5.8) is different from zero. We fix upon such an α. Now choose $u_i^*(x)$, linearly independent of $\bar{u}_i(x)$, so as to satisfy the $n - 2$ conditions

$$U_i(u^*) = 0$$

for i running over all numbers of the set 1, 2, $\cdots$, n except j and α. Then from (5.5) and the boundary conditions already satisfied we have

$$U_j(u^*)V_j(\bar{v}) - U_\alpha(u^*)V_\alpha(\bar{v}) = 0,$$
$$U_j(u^*)V_j(v^*) - U_\alpha(u^*)V_\alpha(v^*) = 0.$$

Since the determinant (5.8) is different from zero it follows from this that $U_j(u^*) = 0$, $U_\alpha(u^*) = 0$, so that $u_i^*(x)$ satisfies (5.1) and (5.6) for $\lambda = \bar{\lambda}$, contrary to the hypothesis that these equations have unique solutions. Hence follows the truth of the second statement in the foregoing theorem.

A value of λ for which the system (5.1), (5.6) [and hence the system (5.2), (5.7)] has a solution will be called a characteristic value. The characteristic value is said to be simple if the solution corresponding to it is unique (except for a constant factor).

THEOREM II. *If*

$$y_1^{(j)}(x), \; y_2^{(j)}(x), \; \cdots, \; y_n^{(j)}(x), \qquad j = 1, 2, \cdots, n,$$

are n linearly independent solutions of (5.1) *for* $\lambda = \bar{\lambda}$, *the condition that* $\bar{\lambda}$ *is a characteristic value is that the determinant*

$$\Delta = \begin{vmatrix} U_1(y^{(1)}) & U_1(y^{(2)}) & \cdots & U_1(y^{(n)}) \\ U_2(y^{(1)}) & U_2(y^{(2)}) & \cdots & U_2(y^{(n)}) \\ \cdot & \cdot & \cdot & \cdot \\ U_n(y^{(1)}) & U_n(y^{(2)}) & \cdots & U_n(y^{(n)}) \end{vmatrix}$$

shall vanish; a characteristic value $\bar{\lambda}$ *of* λ *is simple when and only when some first minor of* Δ *does not vanish.*

If we take the general solution of (5.1) for $\lambda = \bar{\lambda}$ in the form

$$u_i(x) = \sum_{j=1}^{n} c_{ij} y_i^{(j)}(x), \qquad i = 1, 2, \cdots, n,$$

we see that the vanishing of Δ is a sufficient condition that the boundary conditions (5.6) may be satisfied through an appropriate choice of the constants c_{ij}. When some first minor does not vanish this choice is unique (except for a factor which is constant through the set).

THEOREM III. *If* $u_i^{(k)}(x)$ *and* $v_i^{(l)}(x)$ *are solutions of* (5.1), (5.6) *and* (5.2), (5.7), *respectively, the first for* $\lambda = \lambda_k$ *and the second for* $\lambda = \lambda_l$ *and if* $\lambda_k \neq \lambda_l$, *then*

$$\sum_{s=0}^{\infty} \sum_{i=1}^{n} \sum_{j=1}^{n} \psi_{ji}(at^s) u_i^{(k)}(at^s) v_j^{(l)}(at^{s+1}) = 0, \tag{5.9}$$

where $t = q$ *or* q^{-1} *according as* $|q|$ *is greater than or less than unity.*

We have the systems

$$u_i^{(k)}(qx) - u_i^{(k)}(x) = \sum_{j=1}^{n} (\varphi_{ij} + \lambda_k \psi_{ij}) u_j^{(k)}(x),$$

$$v_i^{(l)}(x) - v_i^{(l)}(qx) = \sum_{j=1}^{n} (\varphi_{ji} + \lambda_k \psi_{ji}) v_j^{(l)}(qx) - (\lambda_k - \lambda_l) \sum_{j=1}^{n} \psi_{ji}(x) v_j^{(l)}(qx);$$

multiplying the first by $v_i^{(l)}(qx)$ and the second by $-u_i^{(k)}(x)$, adding member by member, summing as to i from 1 to n and then as to x from a to infinity over the set $a, ta, t^2a, \cdots$, we have a result which (in view of the boundary conditions and the omission of the non-zero factor $\lambda_k - \lambda_l$) reduces to the relation given in the theorem.

Let us now suppose that the problem is set up so that we have the infinitude of characteristic values $\lambda_1, \lambda_2, \lambda_3, \cdots$ and corresponding solutions of the u-problem and of the v-problem; and let us suppose that the first member of (5.9) is different from zero when l and k are equal. Then if given functions $f_i(x)$ $(i = 1, 2, \cdots, n)$ have expansions in the form

$$f_i(x) = \sum_{k=1}^{\infty} c_k u_i^{(k)}(x), \qquad i = 1, 2, \cdots, n,$$

the same coefficients c_k being employed for each of the functions, these coefficients are readily determined as follows: For fixed i multiply both members of the last equation by $\psi_{ji}(x)v_j^{(k)}(qx)$, sum as to i and j from 1 to n, and sum as to x over the set $a, ta, t^2a, \cdots$; employing theorem III we come through readily to the relations*

$$c_k = \frac{\sum_{s=0}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n} \psi_{ji}(at^s)f_i(at^s)v_j^{(k)}(at^{s+1})}{\sum_{s=0}^{\infty}\sum_{i=1}^{n}\sum_{j=1}^{n} \psi_{ji}(at^s)u_i^{(k)}(at^s)v_j^{(k)}(at^{s+1})}, \qquad k = 1, 2, 3, \cdots.$$

Let us illustrate these fundamental expansion formulæ of the q-difference calculus by considering a particularly simple example. We start from the equation

$$u(qx) = \left(1 + \frac{\lambda}{x}\right)u(x), \qquad |q| < 1,$$

and its adjoint

$$\left(1 + \frac{\lambda}{x}\right)v(qx) = v(x).$$

Appropriate solutions of these are the following:

$$u(x) = 1 + \sum_{k=1}^{\infty} \frac{\lambda^k x^{-k}}{(q^{-1} - 1)(q^{-2} - 1)\cdots(q^{-k} - 1)}, \qquad v(x) = \frac{1}{u(x)}.$$

[We may also write

$$u(x) = \prod_{k=1}^{\infty}\left(1 + \frac{q^k\lambda}{x}\right),$$

thus exhibiting $u(x)$ in factored form.] The condition (5.4) now reduces to the simpler form $u(\infty)v(\infty) - u(a)v(a) = 0$. As suitable boundary conditions implying this relation we may take

$$\alpha u(\infty) - u(a) = 0, \qquad v(\infty) - \alpha v(a) = 0,$$

where α is a given non-zero constant. Since $u(\infty) = 1 = v(\infty)$ and $u(a)v(a) = 1$, these boundary conditions require merely that λ shall satisfy the relation $u(a) = \alpha$, or

$$\alpha = 1 + \sum_{k=1}^{\infty} \frac{\lambda^k a^{-k}}{(q^{-1} - 1)(q^{-2} - 1)\cdots(q^{-k} - 1)}.$$

The roots of this transcendental equation in λ are the characteristic values of λ for this special expansion problem. If α is not an exceptional value

* If we use a single equation of order n to replace the n equations of order unity in each system, we shall come through to an expansion problem very closely related formally to that for differential equations in § 4.

for the function of λ in the second member of this equation, the number of such characteristic values is enumerably infinite, so that we have an instance of the case supposed in the preceding paragraph. The simpler expansion formulæ pertaining to the present case may be written down at once by specializing the formulæ of the preceding paragraph.

If we apply to systems (5.1) and (5.2) a limiting process which has become classic through the investigations of Volterra, we are led through to adjoint integro-q-difference equations of the form

$$\begin{aligned} u(qx, \sigma) - u(x, \sigma) &= \int_a^\beta \{\varphi(x, \sigma, \tau) + \lambda\psi(x, \sigma, \tau)\} u(x, \tau) d\tau, \\ v(x, \sigma) - v(qx, \sigma) &= \int_a^\beta \{\varphi(x, \tau, \sigma) + \lambda\psi(x, \tau, \sigma)\} v(qx, \tau) d\tau. \end{aligned}$$

Under appropriate hypotheses the method of successive approximation yields existence theorems for equations of this sort. For them there exists also an expansion problem analogous to that which we have just treated. In the theory of this expansion problem we have the following equations analogous to (5.4) and (5.9):

$$\int_a^\beta \{u(\infty, \sigma)v(\infty, \sigma) - u(a, \sigma)v(a, \sigma)\} d\sigma = 0,$$
$$\sum_{s=0}^{\infty} \int_a^\beta \int_a^\beta \psi(at^s, \tau, \sigma) u^{(k)}(at^s, \sigma) v^{(l)}(at^{s+1}, \tau) d\sigma d\tau = 0, \qquad k \neq l.$$

Then if we have an expansion of the form

$$f(x, \sigma) = \sum_{k=1}^{\infty} c_k u^{(k)}(x, \sigma),$$

the coefficients c_k have the values

$$c_k = \frac{\sum_{s=0}^{\infty} \int_a^\beta \int_a^\beta \psi(at^s, \tau, \sigma) f(at^s, \sigma) v^{(k)}(at^{s+1}, \tau) d\sigma d\tau}{\sum_{s=0}^{\infty} \int_a^\beta \int_a^\beta \psi(at^s, \tau, \sigma) u^{(k)}(at^s, \sigma) v^{(k)}(at^{s+1}, \tau) d\sigma d\tau}, \quad k = 1, 2, 3, \cdots.$$

A similar expansion theory exists for a great variety of generalizations of the q-difference and integro-q-difference equations of this section. It is of importance to have a detailed analysis of the character of the functions which are representable in the form of certain of these expansions.

UNIVERSITY OF ILLINOIS,
April, 1921.